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Locus of boundary crisis: Expect infinitely many gaps

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Boundary crisis is a mechanism for destroying a chaotic attractor when one parameter is varied. In a two-parameter setting the locus of the boundary crisis is associated with curves of homoclinic or heteroclinic bifurcations of periodic saddle points. It is known that this locus has nondifferentiable points. We show here that the locus of boundary crisis is far more complicated than previously reported. It actually contains infinitely many gaps, corresponding to regions (of positive measure) where attractors exist.

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The experimental study of nonlinear dynamical systems is necessarily limited to the observation of attractors and the transitions that they undergo as one or more parameters are varied. Therefore, it is important to study possible transitions also from a more theoretical point of view in order to predict what one expects to see in experiments. For this reason, transitions from steady state or periodic attractors to a regime where attracting chaotic dynamics occurs remains an active field of research.

It is well known that the existence of a chaotic attractor can end abruptly in a boundary crisis when a parameter is varied [1–4]. A boundary crisis occurs when the attractor collides with a periodic orbit on its basin boundary. This periodic orbit is called the crisis orbit. Boundary crisis is a common phenomenon that can be observed in experiments. In many physical experiments chaotic motion is measured as a bounded but seemingly random variation of the amplitude of an underlying oscillation. In a boundary crisis, this chaotic variation suddenly stops and the motion reverts to a regular or even constant-amplitude oscillation. For example, the laser experiment in [5] measures the power spectrum of laser light from an optically injected diode laser. A broad power spectrum indicates that the attractor is chaotic. At a boundary crisis, the output suddenly switches to light with constant intensity. Similarly, in the leaky faucet experiment [6], the time between water drops is measured representing the range of variation for the attractor. In a boundary crisis, the time between drops suddenly changes from irregular to periodic, while the mean drop frequency jumps to a completely different value.

Since boundary crisis is so common, one would expect that slight perturbations in the experimental setup, i.e., small variations of other parameters in the system, do not influence the occurrence of this phenomenon. Indeed, it should be possible to trace the boundary crisis in a plane of two parameters. Such a two-parameter study was done in [7,8], where the authors reported the existence of a piecewise-smooth curve of boundary crises. They found that the curve is only piecewise smooth due to changes in the period of the crisis orbit associated with the boundary crisis. Along a smooth segment of the curve the boundary crisis is organized by, effectively, the same crisis orbit. At a so-called double-crisis vertex the curve is not differentiable, and the boundary crisis switches to a crisis orbit with a different period.

From a mathematical point of view, the boundary crisis is a global change in the dynamics that is caused by a tangency between a stable and an unstable manifold. The chaotic attractor is contained in the closure of this unstable manifold, which is associated with either the crisis orbit or a periodic orbit on the attractor. The stable manifold is always associated with the crisis orbit and forms the basin boundary of the attractor. There are three global phenomena that can occur if two such manifolds become tangent. First, a boundary crisis occurs if the stable manifold is indeed the basin boundary and the closure of the unstable manifold is equal to the attractor at the moment of tangency. Second, if the latter holds, but the stable manifold is not the basin boundary, that is, the crisis orbit lies inside the basin of attraction, then an *interior crisis* occurs and the chaotic attractor suddenly grows in size [4,9]. Third, if the attractor is contained in but is smaller than the closure of the unstable manifold, then a *basin boundary metamorphosis* occurs, where the boundary of the basin of attraction becomes fractal [10–13].

The locus of tangency between a stable and an unstable manifold is a smooth curve in a two-parameter plane. However, the global manifestation of this tangency changes type at points where another tangency curve intersects it. Indeed, at a double-crisis vertex two smooth arcs intersect: one arc changes type from a boundary crisis to an interior crisis, and the other from a boundary crisis to a basin boundary metamorphosis [7,8].

It seems that everything is known about the boundary crisis, but this is actually not the case. By definition, a chaotic attractor implies the existence of infinitely many periodic orbits with stable and unstable manifolds. We report here that, in a two-parameter setting, there are actually infinitely many curves of tangencies between two manifolds, and there are infinitely many intersections between tangency curves. In effect, it is impossible to mark a curve segment that is associated with a boundary crisis. The locus of boundary crisis is interrupted by infinitely many gaps, so that it forms a fractal set.

We consider the Hénon map [14] as the paradigm example of a two-dimensional dissipative map. We use the definition, given in [15],

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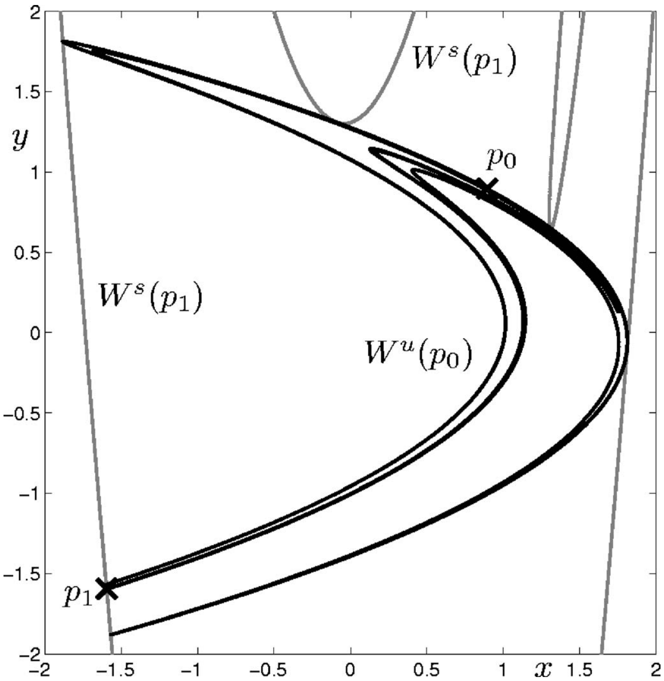

FIG. 1. Manifolds of the Hénon map (1) for $(\alpha,\beta) = (1.426\,921\,2,\ 0.3)$. The unstable manifold $W^u(p_0)$ of the fixed point p_0 is tangent to the stable manifold $W^s(p_1)$ of the fixed point p_1.

$$(x,y) \mapsto (\alpha + \beta y - x^2, y) \tag{1}$$

that is obtained by a parameter-dependent linear scaling of the variables from the standard definition [14]. The parameters α and β remain unchanged in this coordinate transformation so that all crisis and tangency curves in the (α,β) plane are identical for both definitions. The Hénon map has the property that the determinant of the associated Jacobian matrix is constant and equal to $-\beta$. Hence, the map is dissipative for $|\beta|<1$. An attractor exists as soon as $\alpha > -\frac{1}{4}(1-\beta)^2$, and it is believed that the upper bound for α is a (β-dependent) boundary crisis.

Simó [3] has extensively studied the Hénon map for fixed $\beta=0.3$ and varying α. As α increases, a period-doubling route to chaos occurs and a strange attractor exists over a range of α. However, this interval is interspersed with so-called periodic windows where the attractor is not chaotic. A boundary crisis occurs at $\alpha \approx 1.426\,921\,2$ via a tangency between the manifolds of two fixed points [3]. Figure 1 shows the moment of tangency and illustrates how the strange attractor, which is formed by the closure of the unstable manifold $W^u(p_0)$ of the fixed point p_0 touches its own basin boundary, formed by the stable manifold $W^s(p_1)$ of the other fixed point p_1. Hence, p_1 is the crisis orbit.

We focus our attention on the locus of tangency between $W^u(p_0)$ and $W^s(p_1)$ in the two-parameter setting. As α and β are varied continuously, the fixed points p_0 and p_1 vary continuously, and so do the manifolds $W^u(p_0)$ and $W^s(p_1)$. The moment where these two manifolds are tangent forms a curve in the (α,β) plane, which we denote by T_1. Reference [7] reports that T_1 corresponds to a boundary crisis on the segment between the double-crisis vertices $V_0=(2,0)$ and $V_1 \approx (0.973\,005,\ 0.558\,61)$. This segment of T_1, shown in Fig. 2(b), is almost a straight line; see also [[7], Fig. 3]. (The trained eye can observe "ripples" along the curve in Fig. 3 of [7], which are clarified in this paper.)

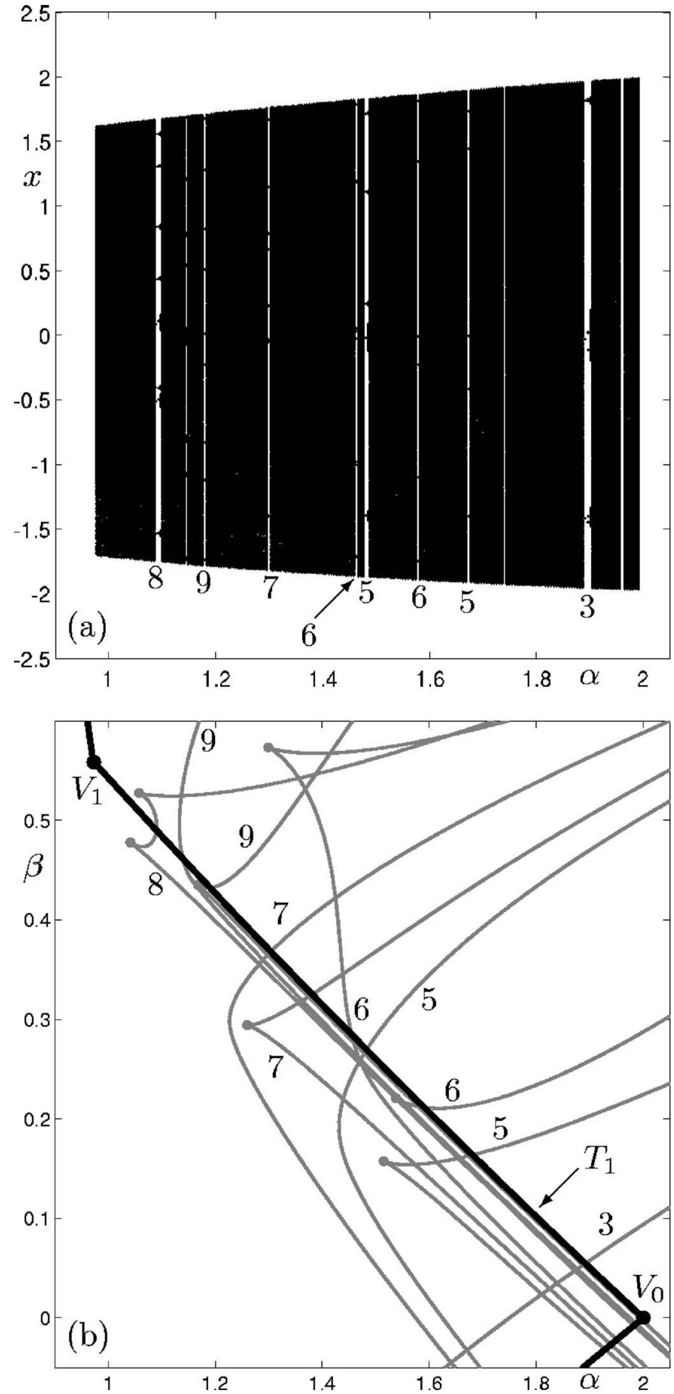

FIG. 2. Bifurcations along the curve T_1 of tangency between $W^u(p_0)$ and $W^s(p_1)$ from the double-crisis vertices V_1 to V_0. The chaotic attractor that exists just to the left of T_1 contains periodic windows (a), each of which is initiated by saddle-node bifurcations that cross T_1 transversely (b). The labels in (a) indicate the periods of the periodic windows and the corresponding bifurcating periodic orbits are indicated in (b).

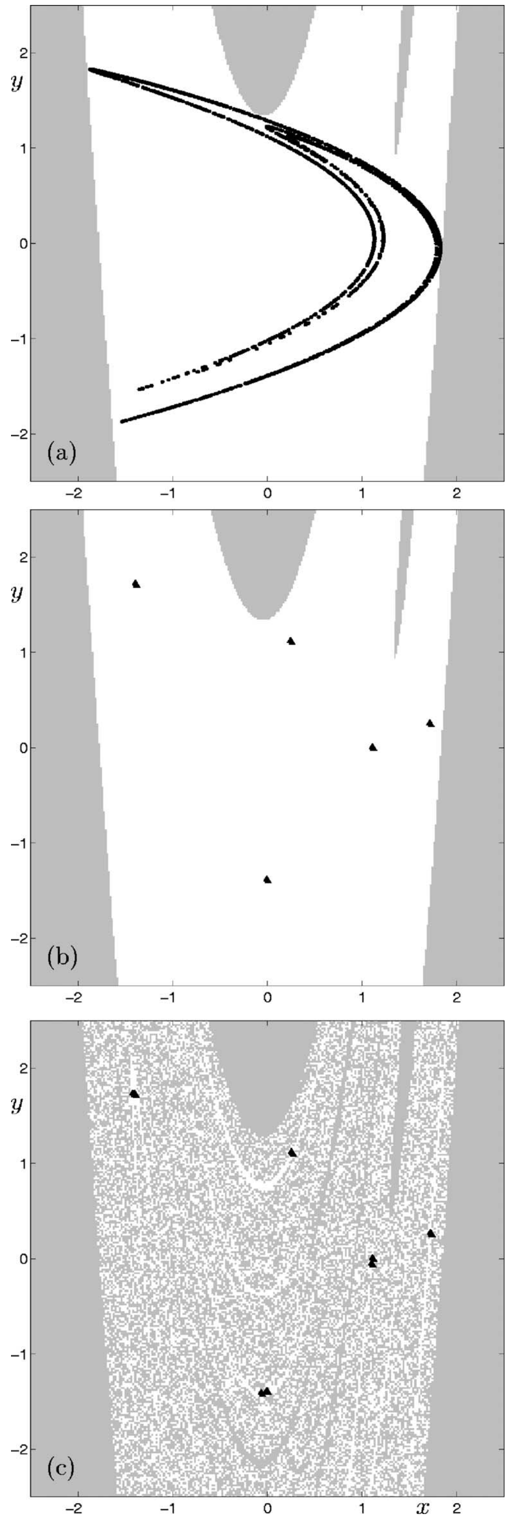

FIG. 3. Basins of attraction with the attractors of the Hénon map (1) for (α,β) near $(\alpha_5,\beta_5)=(1.490\,17,\ 0.265\,53)$. The phase portraits are for $(\alpha_5,\beta_5-0.005)$ (a), $(\alpha_5-0.01,\beta_5)$ (b), and $(\alpha_5,\beta_5+0.005)$ (c).

Figure 2(a) shows a projection onto the (α,x) plane of the attractor just to the left of T_1, that is, just before the boundary crisis, along the curve segment $(\alpha_T-0.01,\beta_T)$, for (α_T,β_T) on T_1 in between V_0 and V_1. The picture clearly shows periodic windows, where, for a range of α, the attractor does not densely fill one large interval of x values. Inside these periodic windows the attractor is, in fact, a k-periodic orbit p_k that undergoes a sequence of period-doubling bifurcations until a chaotic attractor emerges that consists of k pieces. The period k, i.e., the number of pieces, is labeled in Fig. 2(a).

The start of a k-periodic window is a saddle-node bifurcation. We computed curves of saddle-node bifurcations of k-periodic orbits in the (α,β) plane using continuation with CONTENT [16]. Figure 2(b) shows how these curves (in gray for the periods labeled) intersect T_1 transversely. The range of α for the two figures is the same so that one can compare the start of the periodic windows in Fig. 2(a) with the intersections of the saddle-node bifurcation curves with T_1 in Fig. 2(b). Since the attractor is a periodic orbit just to the right of each saddle-node bifurcation curve, each intersection point on T_1 is an *end point* of the boundary crisis. The ripples that one can observe in [[7], Fig. 3] appear to lie exactly in the regions where the periodic windows exist.

The end of each k-periodic window, by which we mean the moment where the attractor becomes one large attractor again, is an interior crisis. The interior crisis is caused by a tangency of the manifolds of the k-periodic saddle orbit that appeared in the saddle-node bifurcation at the start of a k-periodic window. Each of these k-periodic tangencies lies on a curve T_k of k-periodic tangencies in the (α,β) plane. The tangency curves T_k lie almost parallel to the k-periodic saddle-node bifurcation curves, and also intersect T_1 transversely. All such intersections points are double-crisis vertices, that is, to the left of T_1 each curve T_k is an interior crisis, but to the right of T_1 it is a boundary crisis with a crisis orbit of period k. The segment of T_1 in between a k-periodic saddle-node bifurcation curve and the associated tangency curve T_k does not correspond to a boundary crisis; namely, the attractor is relatively small here and far away from its basin boundary. Indeed, close to the saddle-node bifurcation curve, the attractor is not even chaotic. Hence, instead of a boundary crisis, these segments of T_1 instead cause a basin boundary metamorphosis.

In order to illustrate what happens along T_1, we checked the dynamics in a neighborhood of the double-crisis vertex at the end of the first 5-periodic window in Fig. 2(a), approximately at $(\alpha_5,\beta_5)=(1.490\,17,\ 0.265\,53)$. Figure 3 shows from top to bottom the attractors and their basins for parameter values just below, to the left, and above (α_5,β_5), respectively. Figure 3(a) shows the situation for $(\alpha_5,\beta_5-0.005)$, which is as expected for parameter values close to a boundary crisis: the attractor and its basin boundary are similar to the manifolds $W^u(p_0)$ and $W^s(p_1)$ in Fig. 1. For a slightly larger value of α the curve T_1 is reached, the manifolds $W^u(p_0)$ and $W^s(p_1)$ become tangent and a boundary crisis takes place.

Figure 3(b) is for $(\alpha_5-0.01,\beta_5)$, which is inside the first 5-periodic window in Fig. 2(a). While the basin of attraction is virtually the same as in Fig. 3(a), the attractor is a periodic orbit of period 5. Indeed, moving from $(\alpha_5,\beta_5-0.005)$ to $(\alpha_5-0.001,\beta_5)$ is very similar to entering the 5-periodic window from the right in Fig. 2(a): a (reversed) interior crisis occurs followed by a (reversed) period-doubling sequence to a 5-periodic attracting orbit.

The transition from $(\alpha_5 - 0.01, \beta_5)$ to the situation in Fig. 3(c) involves a crossing of T_1. However, no boundary crisis takes place. Figure 3(c) shows that the attractor persists—it is a 10-periodic orbit here—but the basin of attraction has changed dramatically. As predicted, a basin boundary metamorphosis takes place.

The description of the 5-periodic window is typical for all other periodic windows along T_1. Even though we labeled only eight of them in Fig. 2(a), one must expect infinitely many periodic windows that cannot be observed at the scale of the figure. For example, there are two saddle-node bifurcation curves of period 9 and also two of period 7 in Fig. 2(b), but there is only one periodic window each in Fig. 2(a). A zoom of the bifurcation diagram does reveal the other two periodic windows in the parameter region indicated by the intersection points. For each of these possibly infinitely many periodic windows the associated curve segment on T_1 in between the double-crisis vertices V_0 and V_1 causes a basin boundary metamorphosis, rather than a boundary crisis. Hence, each of these segments is a gap on T_1, where this curve is not associated with a boundary crisis. The recursive structure of the periodic windows suggest that the gaps are created in a Cantor-like process, so that the locus of boundary crisis along T_1 is a fractal set.

In summary, the overall structure of the (α, β) plane is drastically different from what is reported in [7,8], where it is suggested that an attractor exists up to a piecewise-smooth curve that ranges, roughly, over $\alpha \in [0.56, 4.2]$; see also [[7], Fig. 3]. We have shown that there are likely infinitely many gaps along the curve segment in between the double-crisis vertices $V_0 = (2,0)$ and $V_1 \approx (0.973\,005,\ 0.558\,61)$, where the boundary crisis is reportedly caused by a tangency T_1 between the manifolds of fixed points [7]. In fact, the Hénon map has infinitely many of such tangency curves that crisscross the (α, β) plane. Gaps in the locus of boundary crisis occur where tangency curves intersect. Hence, it is not correct to think that the locus of boundary crisis forms a single piecewise-smooth curve with a finite number of nondifferentiable points (the double-crisis vertices). The recursive process of removing gaps along tangency curves likely leads to a fractal set as the locus of boundary crisis.

In an experiment, including a numerical one, it is difficult to avoid jumping over smaller gaps when attempting to trace a locus of boundary crisis. As a result a misleading bifurcation diagram may be obtained. We point out that the gaps do not necessarily correspond to attractors with a small basin, although there does seem to be a relationship between the size of the basin and the width of the gap. In [17] a large gap was reported in a three-dimensional dynamical system that appears to be bounded in exactly the same way as discussed here.

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